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LETTER TO THE EDITOR

Liouville surface and new nonlinear integrable equations

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Abstract. New nonlinear equations are obtained by embedding a Liouville surface in a space of larger dimension. Both the cases in which the embedding space is flat and curved are considered and they lead to two different nonlinear equations. Lastly a new determination of the second fundamental form is indicated in which a coupled set of three nonlinear equations are obtained. In each case we also obtain the Lax pair equations.

In recent years a geometrical approach to nonlinear equations has been very successful in yielding new classes of equations and also for obtaining the corresponding Lax pairs. In fact the famous Bäcklund transformation initially originated in differential geometric studies of pseudospherical surfaces (Eisenhart 1909). The whole subject really began with the celebrated paper of Regge and Lund (1976), about the embedding of lower-dimensional surfaces in a higher-dimensional one. Here in this paper we show that it is possible to generate a new class of nonlinear partial differential equations by embedding a surface whose line element is given by

$$ds^2 = du^2/\sin^2\tfrac{1}{2}\omega + dv^2/\cos^2\tfrac{1}{2}\omega \tag{1}$$

where ω is a function of (u, v) in E^3. A surface with such an element was initially studied by Liouville and so it is called after his name. Instead of the present-day differential form approaches, we will follow the traditional tensorial notation.

Let V_n be a Riemannian manifold of dimension n and E_n+1 a space of larger dimension $n+1$. In fact the larger space need not have dimension $(n+1)$ but can have any dimension $m>n$. The larger space may be flat or curved. In the following we have considered both of these cases.

Let us denote the line element of V_n as

$$ds^2 = g_{ij}dx_i dx_j. \tag{2}$$

The simplest and the most transparent form of Gauss–Codazzi equations are reproduced in the Weirstrassian coordinates denoted by Z^α, $\eta^\alpha_{\sigma|}$ where we have followed the notation of Eisenhart (1964).

These equations written in full read:

$$\begin{aligned} z^\alpha_{,ij} &= \sum e_\alpha \Omega_{\sigma|ij}\eta^\alpha_{\sigma|} - K_0 g_{ij} z^\alpha \\ \eta^\alpha_{\sigma|ij} &= -\Omega_{\sigma|il} g^{lm} z^\alpha_{,m} + \sum e_\tau \mu_{\tau\sigma|j}\eta^\alpha_{\tau|} \end{aligned} \tag{3}$$

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where the integrability conditions read:

$$R_{ijkl} = \sum e_\sigma(\Omega_{\sigma|ik}\Omega_{\sigma|jl} - \Omega_{\sigma|il}\Omega_{\sigma|jk}) + K_0(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{4}$$

K_0 = the curvature of the enveloping space,

$$\Omega_{\sigma|ij,k} - \Omega_{\sigma|ik,j} = \sum e_\tau(\mu_{\tau\sigma|k}\Omega_{\tau|ij} - \mu_{\tau\sigma|j}\Omega_{\tau|ik}) \tag{5}$$

and lastly the equation of Ricci written as

$$\mu_{\tau\sigma|j;k} - \mu_{\tau\sigma|k;j} + \sum e_\rho(\mu_{\rho\tau|j}\mu_{\rho\sigma|k} - \mu_{\rho\tau|k}\mu_{\rho\sigma|j}) + g^{lh}(\Omega_{\tau|lj}\Omega_{\sigma|hk} - \Omega_{\tau|lk}\Omega_{\sigma|hj}) = 0. \tag{6}$$

$\mu_{\tau\sigma|j}$ are the components of the torsion and in these equations it should be observed that ρ, σ, $\tau = n+1, \ldots$.

With these words about the formalism let us now consider the case when $m = 3$. Then ρ, σ, τ can only have a value each equal to $n+1 = 3$, that is $n = 2$. In the above equations $\Omega_{\sigma|ij}$ denotes the coefficients of the second fundamental form associated with the surface. In our particular case the integrability conditions can be explicitly written as

$$\frac{\partial D_1}{\partial u} - \frac{\partial D_2}{\partial v} - \frac{D_1}{2E}\frac{\partial E}{\partial u} + \left(\frac{1}{2E}\frac{\partial E}{\partial v} - \frac{1}{2G}\frac{\partial G}{\partial v}\right)D_2 - \frac{1}{2G}\frac{\partial E}{\partial u}D_3 = 0$$

$$\frac{\partial D_3}{2v} - \frac{\partial D_2}{\partial u} - \frac{D}{2E}\frac{\partial E}{\partial v} + \left(\frac{1}{2G}\frac{\partial G}{\partial u} - \frac{1}{2E}\frac{\partial E}{\partial u}\right)D_2 - \frac{1}{2G}\frac{\partial G}{\partial v}D_3 = 0$$

$$\frac{D_1D_2 - D_3^2}{\sqrt{EG}} = -\left[\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial u}\right)\right] \tag{7}$$

where we have set

$$\Omega_{3|11} = D_1 \qquad \Omega_{3|22} = D_2 \qquad \Omega_{3|12} = D_3 \tag{8}$$

and

$$g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

It is now easy to see from equation (1), that is from the values of E, F, G and equation (7), that one possible choice (if the enveloping space is flat) is

$$D_1 = \operatorname{cosec}^2\tfrac{1}{2}\omega \qquad \text{and} \qquad D_2 = \sec^2\tfrac{1}{2}\omega, \qquad D_3 = 0$$

and the only component of the curvature tensor which survives is R_{1212} given also by

$$R_{1212} = \tfrac{1}{2}\left(2\frac{\partial^2 g_{12}}{\partial u\partial v} - \frac{\partial^2 g_{11}}{\partial v^2} - \frac{\partial^2 g_{22}}{\partial u^2}\right) + g_{\alpha\beta}\left[\begin{Bmatrix}\alpha\\12\end{Bmatrix}\begin{Bmatrix}\beta\\12\end{Bmatrix} - \begin{Bmatrix}\alpha\\11\end{Bmatrix}\begin{Bmatrix}\beta\\22\end{Bmatrix}\right]. \tag{9}$$

Equating these two expressions of R_{1212} we get

$$\frac{\partial}{\partial u}\left[\tan^2\tfrac{1}{2}\omega\frac{\partial\omega}{\partial u}\right] - \frac{\partial}{\partial v}\left[\cot^2\tfrac{1}{2}\omega\frac{\partial\omega}{\partial v}\right] = \tfrac{1}{2}\sin\omega \tag{10}$$

the required nonlinear equation for the unknown function $\omega(u, v)$ and it is easy to make contact with the usual formalism if we set $u = x$, $v = t$.

Now the most interesting part of the differential geometric approach is that inverse scattering equations are supplied by equations (3).

In terms of the Weirstrassian coordinates they read

$$\begin{pmatrix} y_+ \\ y_- \\ z \end{pmatrix}_u = \begin{pmatrix} 0 & 0 & \Omega_{3|11} \\ 0 & 0 & \Omega_{3|11} \\ -\frac{1}{2}\Omega_{3|11}g^{11} & -\frac{1}{2}\Omega_{3|11}g^{11} & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \\ z \end{pmatrix} \tag{11}$$

and

$$\begin{pmatrix} y_+ \\ y_- \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Omega_{3|22} \\ 0 & 0 & -\Omega_{3|22} \\ -\frac{1}{2}\Omega_{3|22}g^{22} & \frac{1}{2}\Omega_{3|22}g^{22} & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \\ z \end{pmatrix}. \tag{12}$$

If the enveloping space is curved the nonlinear equations read

$$\frac{\partial}{\partial u}\left[\tan^2\tfrac{1}{2}\omega\frac{\partial\omega}{\partial u}\right] - \frac{\partial}{\partial v}\left[\cot^2\tfrac{1}{2}\omega\frac{\partial\omega}{\partial v}\right] = \tfrac{1}{2}\sin\omega + \frac{K_0}{2\sin\omega}. \tag{13}$$

The corresponding IST equations are modified as

$$\begin{pmatrix} y_+ \\ y_- \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Omega_{3|11} & -K_0 g_{11} \\ 0 & 0 & \Omega_{3|11} & -K_0 g_{11} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\Omega_{3|11}g^{11} & \frac{1}{2}\Omega_{3|11}g^{11} & 0 & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \\ y \\ z \end{pmatrix} \tag{14}$$

and

$$\begin{pmatrix} y_+ \\ y_- \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Omega_{3|22} & -Kg \\ 0 & 0 & -\Omega_{3|22} & K_0 g_{22} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\Omega_{3|22}g^{22} & \frac{1}{2}\Omega_{3|22}g^{22} & 0 & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \\ y \\ z \end{pmatrix}. \tag{15}$$

Lastly let us indicate a different solution of the second fundamental form. We begin by setting

$$D_1 = D_2 \qquad \text{and} \qquad D_3 \neq 0;$$

then equations (7) can be converted to three coupled nonlinear partial differential equations written in the following form.

To simplify the expressions we set

$$D_1 = h(u, v)\cos\omega \qquad D_2 = g(u, v)\tan\tfrac{1}{2}\omega.$$

Then h, g, ω satisfy

$$g_v = hu\cos\omega/\tan\tfrac{1}{2}\omega$$

$$\frac{\partial}{\partial v}[h\cos\omega\exp(-\tfrac{1}{2}\cot^2\tfrac{1}{2}\omega)] = \frac{\partial}{\partial u}(g\tan^2\tfrac{1}{2}\omega)\times\frac{\tan\frac{1}{2}\omega}{\exp(\frac{1}{2}\cot^2\frac{1}{2}\omega)} \tag{16}$$

$$h^2\cos^2\omega - g^2\tan^2\tfrac{1}{2}\omega = -\frac{2}{\sin\omega}\left[\frac{\partial}{\partial u}\left(\tan^2\tfrac{1}{2}\omega\frac{\partial\omega}{\partial u}\right) - \frac{\partial}{\partial v}\left(\cot^2\tfrac{1}{2}\omega\frac{\partial\omega}{\partial v}\right)\right]$$

which form a set of coupled nonlinear partial differential equations associated with the Lax pair

$$\begin{pmatrix} \tilde{y}_+ \\ \tilde{y}_- \\ \tilde{z} \end{pmatrix}_u = \begin{pmatrix} 0 & 0 & h\cos\omega \\ 0 & 0 & g\tan\frac{1}{2}\omega \\ -h\cos\omega/\sin^2\frac{1}{2}\omega & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{y}_+ \\ \tilde{y}_- \\ \tilde{z} \end{pmatrix} \tag{17}$$

and

$$\begin{pmatrix} \tilde{y}_+ \\ \tilde{y}_- \\ \tilde{z} \end{pmatrix}_v = \begin{pmatrix} 0 & 0 & g\tan\frac{1}{2}\omega \\ 0 & 0 & h\cos\omega \\ 0 & -h\cos\omega/\cos^2\frac{1}{2}\omega & 0 \end{pmatrix} \begin{pmatrix} \tilde{y}_+ \\ \tilde{y}_- \\ \tilde{z} \end{pmatrix}.$$

It is also possible to extend this case to an embedding space of constant curvature K_0 and obtain further generalisations of nonlinear equations and Lax pairs.

In this connection it is worth mentioning that as in the usual geometrical approaches the eigenvalue parameter is not there from the very beginning. The invariance of the resulting equations under some transformation in the (x, t) plane (scaling, stretching and so on) is to be exploited for introducing this parameter. This and other related questions will be discussed in a future communication.

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